

**The Role
of the
Definability Concept
in
Finite Model Theory**

R. Gregory Taylor

Source

1. H.-D. Ebbinghaus and Jörg Flum, *Finite Model Theory*, 2nd edition, Springer–Verlag, Berlin, 1999.

Satisfaction Relation

1. Terms and formulas of FO inductively defined to be certain finite symbol strings.
2. Intended meanings assigned to sentential connectives and quantifiers
3. Satisfaction relation $\mathcal{A} \models \phi[\alpha]$ is defined for structures \mathcal{A} , sentences ϕ , and assignments α

Satisfiability

1. Set Φ of sentences is *satisfiable*
2. There exists \mathcal{A} and assignment α
3. $\mathcal{A} \models \phi[\alpha]$
4. For every $\phi \in \Phi$

Equivalence

1. ψ is a consequence of Φ : $\Phi \models \psi$
2. ψ is logically valid if $\emptyset \models \psi$
3. ϕ and ψ logically equivalent if $\phi \leftrightarrow \psi$ logically valid
4. $\Phi \models_{fin} \phi$
5. $\models_{fin} \phi$
6. $\models_{fin} \phi \leftrightarrow \psi$

Characterizability I

1. Characterizable in FOL up to isomorphism
2. Fixed vocabulary τ
3. Finite structure \mathcal{A}
4. Then there exists FO sentence $\phi_{\mathcal{A}}$
5. $\mathcal{B} \models \phi_{\mathcal{A}}$ iff $\mathcal{B} \cong \mathcal{A}$

Characterizability II

1. K a class of finite structures
2. Then \exists a class Φ of FO sentences with
 $K = Mod(\Phi)$
3. K is precisely the class of finite models of
 Φ

FO Axiomatizability

1. K a class of finite structures
2. K axiomatizable in FOL if \exists single sentence ϕ such that $K = \text{Mod}(\phi)$
3. New paradigm of mathematical definability (expressive power of a logic, e.g., FOL)

Quantifier Rank

1. Maximum nesting of quantifiers
2. Atomic ϕ, ψ have q. r. 0
3. $\phi \wedge (\exists z)\psi$ has q. r. 1
4. $(\forall x)[\phi \wedge (\exists y)(\exists z)\psi]$ has q. r. 3

m -Equivalence

1. \mathcal{A} and \mathcal{B} finite structures
2. $m \in \mathcal{N}$
3. $\mathcal{A} \equiv_m \mathcal{B}$ if satisfy same FO sentences of quantifier rank $\leq m$

Theorem on Nonaxiomatizability

1. K a class of finite structures
2. Suppose $\forall m \in \mathcal{N} \exists$ finite structures \mathcal{A} and \mathcal{B} such that $\mathcal{A} \in K \wedge \mathcal{B} \notin K \wedge \mathcal{A} \equiv_m \mathcal{B}$
3. Then K not axiomatizable in FOL, and vice versa.

Partial Isomorphisms

1. Structures \mathcal{A}, \mathcal{B} over vocabulary τ
2. Any structure preserving bijection between a subset of A and a subset of B
3. Example: $0 \rightarrow 0, 5 \rightarrow \pi, 10 \rightarrow \pi + 1$ between $\langle \mathcal{N}, < \rangle$ and $\langle \mathcal{R}, < \rangle$

Finite Ehrenfeucht Games (1961)

1. Provide game-theoretic characterization of relation $\mathcal{A} \equiv_m \mathcal{B}$
2. \mathcal{A} and \mathcal{B} finite structures over common vocabulary τ
3. Two players *spoiler* and *duplicator*
4. Spoiler's (duplicator's) goal is to exploit structural differences (similarities) between \mathcal{A} and \mathcal{B}
5. Finite number m gives number of moves (fixed in advance and known to both players)

Finite Ehrenfeucht Games (cont.)

1. Each move involves spoiler choosing single element of domain A or of domain B ; duplicator then chooses element of B or A
2. Duplicator wins play if after m moves the chosen elements comprise a partial isomorphism between \mathcal{A} and \mathcal{B} ; otherwise, spoiler wins play
3. Spoiler (duplicator) has *winning strategy* or *wins game* $G_m(\mathcal{A}, \mathcal{B})$ if can win any play of game
4. If $\mathcal{A} \cong \mathcal{B}$ then duplicator wins game $G_m(\mathcal{A}, \mathcal{B})$ for arbitrary m

Ehrenfeucht's Theorem

1. Vocabulary τ , $m \geq 0$
2. Finite structures \mathcal{A}, \mathcal{B}
3. Duplicator wins $G_m(\mathcal{A}, \mathcal{B})$ iff $\mathcal{A} \equiv_m \mathcal{B}$
4. Attainment of goal of game-theoretic characterization of $\mathcal{A} \equiv_m \mathcal{B}$
5. Actually special case of theorem

m -Isomorphism

1. $\mathcal{A} \cong_m \mathcal{B}$ if there exists $(I_j)_{j \leq m}$
2. Each I_j is nonempty set of partial isomorphisms from \mathcal{A} to \mathcal{B}
3. (Forth property) For every $j < m, p \in I_{j+1}, a \in A$, there exists $q \in I_j$ with $q \supseteq p$ and $a \in \text{Domain}(q)$
4. (Back property) For every $j < m, p \in I_{j+1}, b \in B$, there exists $q \in I_j$ with $q \supseteq p$ and $b \in \text{Range}(q)$

Fraïssé's Theorem

1. Structures \mathcal{A}, \mathcal{B} over vocabulary τ
2. $\mathcal{A} \cong_m \mathcal{B}$ iff $\mathcal{A} \equiv_m \mathcal{B}$

Application

1. τ the empty vocabulary
2. Structures \mathcal{A}, \mathcal{B} over τ are just nonempty sets
3. If $|\mathcal{A}|, |\mathcal{B}| \geq m$ then $\mathcal{A} \cong_m \mathcal{B}$
4. Let \mathcal{A}_m be τ -structure of size m
5. If $\mathcal{A}_m \in \text{EVEN}[\tau]$ then $\mathcal{A}_{m+1} \notin \text{EVEN}[\tau]$
6. But $\mathcal{A}_m \cong_m \mathcal{A}_{m+1}$
7. So $\text{EVEN}[\tau]$ not axiomatizable in FOL

Applications to Graph Theory

1. Class of finite connected graphs not axiomatizable in FOL
2. Intuitive argument: No way to say "there is a path between any two nodes"
3. Formal argument: Use Theorem on Non-axiomatizability (m -equivalence)

Second-Order Logic

1. FOL plus n -ary relation variables and quantifiers over them
2. $\text{EVEN}[\tau]$ axiomatizable in SOL by "there is an equivalence relation all of whose equivalence classes have exactly two elements"

Monadic Second-Order Logic

1. n -ary relation (set) variables with $n = 1$ only
2. $\mathcal{A} \equiv_m^{MSO} \mathcal{B}$ if \mathcal{A}, \mathcal{B} satisfy same MSO sentences of q. r. $\leq m$

MSO and Games

1. Finite structures \mathcal{A}, \mathcal{B}
2. Spoiler and then duplicator
3. May choose element a in A or element b in B
4. May choose subset P of A or subset Q of B

MSO and Games (cont.)

1. Duplicator wins play iff sequence of pairings of domain elements is partial isomorphism between $\langle \mathcal{A}, P_1, P_2, \dots \rangle$ and $\langle \mathcal{B}, Q_1, Q_2, \dots \rangle$
2. Duplicator wins game $G_m^{MSO}(\mathcal{A}, \mathcal{B})$ iff $\mathcal{A} \equiv_m^{MSO} \mathcal{B}$

Application

1. Not hard to see that the class of finite, connected graphs is axiomatizable by a sentence of the form $(\forall P)\psi$, where P is a monadic variable (CONN is $M\Pi_1^1$ -definable)
2. Can prove (using Hanf's Theorem) that CONN not axiomatizable by sentence of form $(\exists P_1)(\exists P_2)\dots(\exists P_r)\psi$ (CONN not $M\Sigma_1^1$ -definable)

Infinitary Logics

1. $L_{\infty\omega}$ like FOL but with arbitrarily long disjunctions and conjunctions
2. $L_{\omega_1\omega}$ like FOL but with countably long disjunctions and conjunctions
3. Semantics extends that of FOL
4. $\mathcal{A} \models \Psi$ iff $\mathcal{A} \models \psi$ for some $\psi \in \Psi$

Expressivity of $L_{\omega_1\omega}$

1. For any τ , class $\text{EVEN}[\tau]$ is axiomatizable
2. Any class K of finite structures axiomatizable by FO characterizability of any given finite model
3. CONN axiomatizable

A Notion of Equivalence

1. Two structures \mathcal{A}, \mathcal{B} are $L_{\infty\omega}$ – *equivalent* if satisfy same $L_{\infty\omega}$ -sentences
2. $\mathcal{A} \equiv^{L_{\infty\omega}} \mathcal{B}$

Infinite Games

1. Goal characterize $L_{\infty\omega}$ -equivalence
2. Like $G_m(\mathcal{A}, \mathcal{B})$ but now infinitely many moves
3. Each move involves choice of e_i from A and f_i from B
4. Duplicator wins play if every $e_1, e_2, \dots, e_n \rightarrow f_1, f_2, \dots, f_n$ is partial isomorphism for all n

$L_{\infty\omega}$ -Equivalence and Games

1. Duplicator wins game $G_{\infty}(\mathcal{A}, \mathcal{B})$ iff $\mathcal{A} \equiv^{L_{\infty\omega}} \mathcal{B}$

Finite-State Automata

1. Input alphabet Σ (typically $\{a, b\}$)
2. Finite set Q of states
3. Unique initial state $q_0 \in Q$
4. Terminal state set $F \subset Q$
5. Transition mapping $\delta : Q \times \Sigma \rightarrow Q$
6. $\delta(q_5, b) = q_6$

Characterization

1. Language L is FSA-acceptable iff L definable in MSO
2. What does ' L definable in MSO' mean?

Word Model \mathcal{B}_w

1. Assume word w over alphabet Σ
2. Vocabulary $<$ and monadic P_s for all $s \in \Sigma$
3. Domain $B_w = \{1, 2, \dots, |w|\}$
4. $k \in P_s^{\mathcal{B}_w}$ iff s is k th symbol of w

Example: Word Model A for $abaaab$

1. Vocabulary \langle, P_a, P_b
2. Domain $A = \{1, 2, 3, 4, 5, 6\}$
3. $P_a^A = \{1, 3, 4, 5\}$
4. $P_b^A = \{2, 6\}$

Definability in MSO

1. L definable in MSO if \exists MSO sentence ϕ with $Mod(\phi) = \{\mathcal{B}_w | w \in L\}$
2. Second-order paradigm of definability
3. Language L is FSA-acceptable iff L definable in MSO
4. Proof by induction on the complexity of regular expressions, e.g., $a(a|b)^*b$, for FSA-acceptable languages

Application I

1. Bipartite graph $G = (V, E)$
2. Balanced if bipartite and $|X| = |V \setminus X|$
3. BAL not axiomatizable in MSO

Application II

1. Hamiltonian circuit: closed path visiting every vertex other than start-stop exactly once
2. HAM not axiomatizable in MSO

Application III

1. Clique: set of nodes any two of which connected by edge
2. CHS = graphs possessing clique containing at least half the vertices in V
3. CHS not axiomatizable in MSO

First-Order Definability

1. L is *plus free regular* if denoted by regular expression with no occurrence of $+$
2. L p. f. r. iff L definable in FO

Noncounting Languages

1. L over $\{a\}$ *noncounting* if $\exists k \geq 1$ such that $(\forall y \in A^+) xy^k \in L$ iff $xy^{k+1} \in L$
2. Result: L definable in FO iff L noncounting

Descriptive Complexity Theory

1. Goal: give logical descriptions of complexity classes
2. E.g., K a class of structures and $K \in P$
3. Then $K = Mod(\phi)$ for some $\phi \in IFP(FO)$
4. $IFP(FO)$ captures K (new second-order paradigm)

Extensions of FOL

1. Inflationary Fixed-Point Logic (IFP)
2. Partial Fixed-Point Logic (PFP)
3. Deterministic Transitive Closure Logic (DTC)
4. Transitive Closure Logic (TC)

Fixed Points

1. M a set
2. Function $F : \wp(M) \rightarrow \wp(M)$
3. Induces sequence $\emptyset, F(\emptyset), F(F(\emptyset)), \dots$
4. Alternatively: F_0, F_1, F_2, \dots
5. Fixed point $F_{n_0} = F_{n_0+1} = F_\infty$ may exist
6. F inflationary if $F_n \subseteq F_{n+1}$ for all $n \geq 0$

Fixed-Point Operations from Formu

1. FO formula $\phi(x_1, \dots, x_k, X)$, k -ary X
2. τ -structure \mathcal{A}
3. Induces $F^\phi : \wp(A^k) \rightarrow \wp(A^k)$
4. $F^\phi(R) := \{\langle a_1, \dots, a_k \rangle \mid \mathcal{A} \models \phi[a_1, \dots, a_k, R]\}$

Example

1. Graph $\mathcal{G} = \langle V, E \rangle$
2. Formula $\phi_0(x, y, X) := Exy \vee \exists z (Xxz \wedge Ezy)$
3. Induces $F_0^\phi = \emptyset, F_1^\phi = E, F_2^\phi = \{(a, b) \mid d(a, b) \leq 2\}, \dots$
4. Inflationary and $F_\infty^\phi = \{(a, b) \mid a \text{ connected to } b\}$
5. Picture:

IFP Logic

1. Extend syntax of FOL
2. Given ϕ , form $[\text{IFP}_{\bar{x}, X} \phi] \bar{t}, \bar{x}, \bar{t}$ both k -ary

Axiomatizability in a Logic

1. K a class of τ -structures
2. \mathcal{L} a logic
3. K axiomatizable in \mathcal{L} if \exists sentence $\phi \in \mathcal{L}[\tau]$ with $K = \text{Mod}(\phi)$
4. CONN axiomatizable in FO(IFP) by $\forall x \forall y (\phi_0(x, y) \vee \exists z (Xxz \wedge Ezy))$

Comparing Logics

1. Vocabulary τ
2. $\mathcal{L}_1 \leq \mathcal{L}_2$ if $\forall \phi \in \mathcal{L}_1[\tau] \exists \psi \in \mathcal{L}_2$ with $Mod(\phi) = Mod(\psi)$
3. \mathcal{L}_2 at least as expressive as \mathcal{L}_1
4. $\mathcal{L}_1 \equiv \mathcal{L}_2$
5. $\mathcal{L}_1 < \mathcal{L}_2$
6. $IFP(FO) \equiv PFP(FO)$

(Deterministic) Transitive Closure

1. R a binary relation over set M
2. $TC(R) := \{(a, b) | \exists n > 0, e_0 = a, e_1, \dots, e_n = b \text{ such that } \forall i < n, e_i R e_{i+1}\}$
3. $DTC(R) := \{(a, b) | \exists n > 0, e_0 = a, e_1, \dots, e_n = b \text{ such that } \forall i < n, e_{i+1} \text{ is unique } e \text{ with } e_i R e_{i+1}\}$

TC(FO) and DTC(FO)

1. Extend syntax of FOL
2. Given formula ϕ expressing binary relation, form new formula $[TC_{x,y}\phi]st$
3. Denotes pairs (s, t) such that \exists a ϕ -path from s to t
4. Given formula ϕ expressing binary relation, form new formula $[TC_{x,y}\phi]st$
5. Denotes pairs (s, t) such that \exists a ϕ -path from s to t

TC(FO) and DTC(FO) (cont.)

1. CONN axiomatizable in TC(FO)
2. Also $\text{DTC(FO)} \leq \text{TC(FO)}$

A Technical Point

1. K a class of structures
2. $K_m := \{\mathcal{A} \mid \mathcal{A} \in K \wedge |\mathcal{A}| \geq m\}$
3. K axiomatizable in logic \mathcal{L} iff K_m axiomatizable in \mathcal{L}
4. Idea: structures of size below m axiomatizable in FOL
5. Frequently possible to focus upon “sufficiently large” structures

Complexity Classes

1. Want to associate logics with complexity classes
2. Example: IFP(FO) captures PTIME

Turing Machines

1. Finite set of state Q
2. Initial state q_0
3. Set of accepting states F^+
4. Set of rejecting states F^-
5. Two-way infinite tape (read-write head)
6. Transition mapping δ
7. Example $\delta(q_2, a) = \{(R, q_3), (b, q_4)\}$

Word Acceptance/Rejection

1. Word w over alphabet Σ
2. Machine M accepts w if started reading leftmost symbol then terminates in state $q \in F^+$
3. Machine M rejects w if started reading leftmost symbol then terminates in state $q \in F^+$

Language Acceptance/Decidability

1. Language L over alphabet Σ
2. L Turing acceptable if $\exists M$ that accepts all and only words $w \in L$
3. L Turing decidable if $\exists M$ that accepts all words $w \in L$ and rejects all $w \notin L$

Famous Complexity Classes

1. PTIME = class of languages accepted by deterministic TMs computing in time polynomial in the length of input w
2. NPTIME = class of languages accepted by nondeterministic TMs computing in time polynomial in the length of input w
3. PSPACE = class of languages accepted by deterministic TMs computing in space polynomial in the length of input w
4. NPSPACE = class languages accepted by nondeterministic TMs computing in space polynomial in the length of input w
5. PSPACE and NPSPACE

Structures as Inputs

1. Inputs to Turing Machines
2. Multiple tapes (including worktapes)
3. First input tape contains $|A|$
4. One input tape for binary representation of extension over A of each k -ary R
5. Input tapes for denotation of constants

Structure Acceptance

1. Structure K
2. Machine M accepts K if started reading binary representation of K then terminates in state $q \in F^+$

Complexity Classes of Structures

1. Class K of structures \in PTIME if \exists deterministic M computing in polynomially bounded time that accepts all and only members of K
2. NPTIME
3. PSPACE
4. NPSPACE
5. LOGSPACE
6. NLOGSPACE

IFP(FO) and PTIME

1. K a class of structures
2. $K \in \text{PTIME}$ iff K axiomatizable in IFP(FO)
3. IFP(FO) captures PTIME

Other Results

1. PFP(FO) captures PSPACE
2. TC(FO) captures NLOGSPACE
3. DTC(FO) captures LOGSPACE
4. Σ_1^1 captures NPSPACE



1.

2.

3.

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1.

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