SUPPLEMENT TO ZERMELO'S ANALYSIS OF 'GENERAL PROPOSITION'

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Abstract. This document contains remarks pertaining to issues discussed in R. Gregory Taylor, 'Zermelo's analysis of "general proposition" ', *History and Philosophy of Logic* **30** (2009) 141–55 (henceforth ZAGP).

Contents

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§1. Introduction.

1.1. Regarding Zermelo's protégé Marvin Farber. Phenomenologist Marvin Farber (1901–80) is best-known as founding editor of *Philosophy and Phenomenological Research*, the successor to Husserl's journal. Farber spent his entire career at the University of Buffalo (now the State University of New York at Buffalo) with the exception of three years as chairperson of the Department of Philosophy at the University of Pennsylvania. Farber's students included philosopher Wilfred Sellars and historian Richard Hofstadter.

Marvin Farber studied music—he was a violinist—as an undergraduate at the University of Buffalo before transferring to Harvard, where he studied philosophy, graduating in 1922. He then commenced graduate studies in Harvard's Department of Philosophy, receiving a Ph.D. in 1925 for a dissertation published a few years later as *Farber 1928*. Between 1923 and 1927 Farber, recipient of a Harvard Traveling Fellowship, spent several semesters at Freiburg and other German universities, where he took courses with Heidegger, Husserl, and Heinrich Rickert. We do not know when or under what circumstances Marvin Farber came to know Zermelo, but they were meeting with some regularity by April 1924. Farber did take two mathematics courses with Zermelo at the University of Freiburg during the academic year 1926–27, but by that time their relationship was already firmly established. Farber never took a course in logic or set theory with Zermelo because Zermelo taught such courses at Freiburg only after Farber's 1927 departure (see *Ebbinghaus with Peckhaus 2007*, pp. 143–44). For more regarding Zermelo's relationship with Farber see *Ebbinghaus with Peckhaus 2007*.

1.2. Regarding materials in the Marvin Farber archive. The Marvin Farber Papers on Philosophy and Phenomenology (22/5F/768) at SUNY Buffalo Archives house a variety of materials pertaining to Zermelo's work from the 1920s on. These include all of the following. (Just (1)–(3) are mentioned in ZAPG.)

1. Farber's notes (folder 33.10), in English and German, of conversations with, and occasional lectures by, Zermelo concerning the philosophy of logic and mathematics. The notes run to over 200 hand-written pages. Although almost none are dated, notes do exist from a 'Lecture of Professor Zermelo—19.2.1924/Freiburg im Breisgau Math. Gesellschaft' as well as notes from 'Talks with Zermelo' dated 15$^{\text{th}}$ April, 24$^{\text{th}}$ April, 8$^{\text{th}}$ June, 9$^{\text{th}}$ June, and 10$^{\text{th}}$ June 1924.

2. Farber's outlines (*ibid.*) of an envisioned book, to be co-authored in English by Zermelo and Farber, and entitled initially *Logic* and later *The Foundations of Logic: Studies Concerning the Structure and Function of Logic*. One outline is hand-written and entitled 'Notes for "Logic" (Z.–F.) (Freiburg)'. We are guessing that this first outline was written during 1926 at one of Farber's first meetings with Zermelo. The second outline is a typescript bearing the longer title. Circumstantial evidence suggests that it was most likely created in Buffalo during 1927. There are 35 chapter titles in both outlines. Folder 33.10 contains notes for Chapters I (The Nature of Logic and Its Problems), II (The Nature of Judgment), III (Assertion and Negation), VI (Theory of Objects), VII (Universal and Particular Judgments), XIII (Fundamental Concepts of Mathematics and Deduction), and XVI (Induction and Probability). These notes are clearly distinguished by chapter numbers and titles and by page numbers within each chapter. In addition, there are many pages marked 'Topics for discussion'. The latter are definitely part of the book project, despite no chapters being indicated.
3. A 34-page, double-spaced, typescript (*Farber 1927*) of an initial draft of Chapters I–III of the book.
4. Farber's notes (folder 33.11) from two mathematics courses taught by Zermelo during the academic year 1926–27. One course is Foundations of Arithmetic, and the other is Complex Variables ('Einführung in die Funktionentheorie').
5. Farber's own lecture notes (folder 28.12) from courses at the University of Buffalo during the period 1927–38 based on (1) and including Farber's unpublished English translation of *Zermelo 1935*.
6. Farber's 200-page typescript *The Nature of Logic and Its Fundamental Principles* (29.4), which does not owe anything to Zermelo on the face of it. It seems that this work was never published.

We assume that the notes described in (1)–(2) are essentially Zermelo's dictation. They are written in a mixture of English and German, often within a single sentence, suggesting that Farber is sometimes writing fast. Pages 18–19 of a chapter entitled 'Deduktion' describe an 11-part, symbol-laden definition of *System*. These two pages only are presented in a very careful German and may have been copied from the blackboard. Otherwise, phrases such as 'Z thinks' and 'according to Z' occur occasionally as well as concepts and formulations that bear Zermelo's unmistakable stamp. In addition, our dictation assumption is in accord with a certain German academic model and with the fact that Zermelo is, after all, Farber's senior by 30 years and a mathematician of considerable standing. (What could the 24-year-old Farber have contributed in any case?)

The joint project is mentioned by Farber in two letters written from Buffalo during 1927 that are now found in Zermelo's archive in Freiburg. One refers specifically to Farber's working out of the chapter on 'Meaning', which would be a reference to Chapter XIV entitled 'Meaning and Indication'. From Zermelo's side there is a first reference to the co-authored book in a letter to Farber from mid-1927.

> Could you not merely send some portion of the logic manuscript that I might look it over in the coming period? During the semester one is hard pressed to get to such things. But just now I would have had a little time.[1]

By late 1927, Zermelo appears to be giving up.

> I am pleased to hear that things are going well for you in your new position [at the University of Buffalo] and that you are making gradual progress in the working out of your—I can hardly continue to say 'our'—project.[2] (from letter of 26th December 1927)

However, two final postcards show Zermelo attempting, once again, to coax Farber into sending the manuscript.

> And how are things going with working out the details of *Logic*, of which I would so much have liked to see the first draft.[3] (from postcard dated 22th March 1928)
>
> I would also like to have first seen some draft of your work on logic in order to determine whether some further collaboration would be feasible and promising. If you send me something before August, I could work on it during the vacation, and if you then should come here during the summer of 1929 (perhaps with Sidney and Norma), the foundation for further work would be laid.[4] (from postcard dated 28th April 1928)

The reference here is to Sidney Farber, brother of Marvin and newly married, whom Zermelo had come to know, through Marvin no doubt,

[1] The German original reads: 'Könnten Sie nur nicht schon etwas logisches Manuskript zur vorläufigen Durchsicht schicken? Im Semester komme ich doch nur wenig dazu. Jetzt hätte ich gerade ein wenig Zeit.'

[2] Original German: 'Es freut mich, daß es Ihnen zu Ihrer neuen Tätigkeit gut geht und Sie auch mit der Ausarbeitung Ihres (ich darf wohl kaum mehr sagen „unseres") Werkes allmählich vorwärts kommen.'

[3] Original German: 'Und was macht Ihre *Logik*-Ausarbeitung, von der ich doch so gern einmal eine Probe gesehen hätte?'

[4] German original: 'Auch möchte ich zunächst einmal eine Probe Ihrer Logik-Darstellung gesehen haben, um mir klar zu werden, ob eine *gemeinsame* Weiterarbeit hier wirklich angängig und erfolgversprechend sein würde. Wenn Sie mir bis zum August etwas schicken, so könnte ich mich in den Ferien damit beschäftigen und wenn Sie dann im nächsten Sommer 1929 event. (vielleicht mit Sidney und Norma) herkämen, so wäre doch schon der Grund für fruchtbare Weiterarbeit gelegt.'

and whom Zermelo mentions affectionately throughout their correspondence. Incidentally, Sidney Farber (1903–73) had an illustrious career as a pediatric pathologist, and Harvard's Dana–Farber Cancer Institute is world-renowned.

We attribute *Farber 1927* to Farber alone. Doubtless he alone wrote it. Moreover, their correspondence (25.25), which ended in 1929, makes clear that Farber never sent *1927* to Zermelo despite emphatic appeals, and no copy of it is to be found in the Freiburg archive. On the other hand, attribution to Farber alone is a very conservative choice: *1927* follows the relevant portions of (1) very closely, and we think that Farber's intellectual input is nil.

We mention in passing that two of Farber's published articles from the late twenties are based in large part, but not entirely, on (1). Regarding *Farber 1927*, Farber writes from Buffalo on 10$^{\text{th}}$ April 1928 as follows.

> During Christmas week I attended the Chicago meeting of the American Philosophical Association and read a paper entitled "Theses Concerning the Foundations of Logic." The paper will be published. In it I acknowledge indebtedness to you, which does not commit you to responsibility for the statements therein contained since you have not yet seen the paper; and on the other hand it will be clear when our work appears that the ideas belong to both of us.

Despite the third sentence here, *Farber 1929* as published contains no acknowledgement of Zermelo's contribution, although there is a gratuitous reference to Zermelo's 1908 axiomatisation. Essentially the same is true of *Farber 1930*: apparent use of Zermelo's ideas from (1) without acknowledgement together with a throw-away reference to *Zermelo 1908*. Despite this, we are not inclined to judge Farber harshly. He had much more material from Zermelo that he might have used but never did. Further, we are very much in Farber's debt: he saved everything, and it is all there in Buffalo—hundreds of pages. Zermelo's own archive is rich in its own way, but there is almost nothing there from the twenties. For that period our only source is Farber.

Questions regarding the dating of *Zermelo 1921* are raised in *Van Dalen and Ebbinghaus 2000*. Briefly, the authors suggest that *Zermelo 1921* might in fact come from a much later period. Since *Farber 1929* is based to some extent on Farber's notes of his conversations with Zermelo, it seems possible that its title has the same source and is derived from the title of *Zermelo 1921*, assuming that Farber was aware of this document or at least its contents. If this is substantially correct, then *Zermelo 1921* is indeed from the twenties.

1.3. Regarding the unanalysability of the fundamental concepts (or 'categories') of logic, mentioned on page 2 of ZAGP.

We quote a characteristic passage—from Farber's notes for Chapter XIII, entitled 'Classes', of *Farber 1927*. (Frege's definition of 'number' by abstraction is an instance of what Zermelo means by 'invariant formation'—an important source of new mathematical concepts according to Zermelo.)

> *Invariant*:—A logical fundamental concept—not definable in the usual sense. The actus can be displayed or classified, but it cannot be defined under an *Oberbegriff*. A logical fundamental operation is the making of an invariant. It may not be led back further, or analysed further. Thus *relations*, *classes* are defined (but classes can also be defined otherwise).

Zermelo's point is that the concepts of relation and class, unlike that of invariant, are not *fundamental* concepts of logic. On the other hand, *they are concepts of logic* according to him, which indicates that, even at the point of his conversations with Farber, Zermelo's conception of logic is closer to Frege and Russell than to Hilbert and Ackermann, say.

1.4. Regarding the fundamental concepts of assertion, negation, and generality, mentioned on page 2 of ZAGP. One would expect Zermelo to hold that assertion and negation, as fundamental logical concepts, are unanalysable. Yet in *Farber 1927* he and Farber present a philosophical analysis of both, in terms of *substrates*, as part of their *Isolation Theory of Judgement*. (Judgements effect limitations within substrates, which are contexts giving rise to judgements.) Incidentally, assertion, negation, and generality all figure in Kant's Table of Categories of the Understanding.

1.5. The concept of substrate. This concept is mentioned in §1 of ZAGP. Although it makes no appearance in his published writings, the notion is one that Zermelo emphasizes in the period around 1924, and it is the subject of his lecture to the Freiburg Mathematical Society in February of that year. It is fair to say that the substrate concept is the direct antecedent of the later concept of a system of propositions.

> The best (most completely expressed) examples of substrates are illustrated by mathematics. The propositions which may be derived from a substrate constitute a system, which is purely deductive in the case of mathematics. (*Farber 1927*, p. 13])

Zermelo's Principle of Substrates, asserting that the Laws of Noncontradiction and Excluded Middle govern any substrate, serves as point of departure for his version of Hilbert's non ignorabimus.

§2. Systems of propositions. In an earlier period, propositions were called *Sätze an sich*—by Bolzano in particular—but Zermelo does not employ this term. In *Tractatus* 4.46–4.4661 the term *Satz* is reserved for contingent propositions. (But the discussion indicates that Wittgenstein regards his terminology as nonstandard.) Certainly Zermelo uses *Satz* for noncontingent propositions in both *Zermelo 1931* and *1935*.

The example concerning Julius Cæsar comes from Farber's notes for Chapter VI, entitled 'Objects', of *The Foundations of Logic*.

2.1. More on substrates. Apparently, for both Russell and Zermelo, propositions, themselves timeless, may have mutable constituents, which means that propositions are like sets in a certain sense. Zermelo's propositions ('judgement-meanings' in *Farber 1927*), like Russell's, are structured, nonlinguistic entities.

> At the basis of every judgment there is a general system of objects with their relationships. This is given "before the judgment," and has been called a substrate. By means of the judgment a character is attributed to the given substrate. ... The substrate forms the subject, and the selected character founds the predicate of the proposition. The predicate may be called a *propositional function*, or a proposition with a variable, to the extent to which the proposition can be true or false for the variable substrate. (*Farber 1927*, p. 20)

Elsewhere in *Farber 1927* substrates are referred to as "contexts giving rise to judgment" (see footnote 3) and in a mathematical setting a substrate comprises a given nonempty domain $\mathfrak{D}$ together with a collection of explicitly given relations over $\mathfrak{D}$. The proposition $R^*(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$, whereby domain elements $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ fill positions $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ within (possibly complex) relation $R^*(\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$, involves attribution of the character founding $R^*(\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ to $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$.

Our reading is, of course, somewhat at odds with the quoted passage from *Farber 1927*. In particular, it is the substrate that is stated to be the subject of the proposition, and the predicate is attributed to it. If the substrate is taken to be domain $\mathfrak{D}$ alone, then it seems possible to hold that, in the proposition $R^*(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$, an attribution is being made, in the first instance, to a certain k-tuple of elements of $\mathfrak{D}$ and thereby, in an extended sense, to domain $\mathfrak{D}$ itself. (When promotions or salary increases are awarded 'to the faculty', the recipients are only those faculty members who have not attained top rank or some salary cap.) If, on the other hand, it is the system of objects $\mathfrak{D}$ *plus relationships* that constitutes the substrate, which is what the first sentence of the quoted passage asserts, then the notion of attribution must be adjusted yet further. The latter reading is supported by 'Notes on the Foundations of Mathematical Logic (Discussed with Zermelo in 1924)' (Farber archive [folder 33.10]), wherein Farber writes that 'a substrate is a system of relations between the elements of a domain'. Farber's brief discussion in *Farber 1928* is yet more at odds with our reading: Zermelo's substrates are there said to be interpretations of mathematical structures (see also *Farber 1930*).

2.2. Comparison of Russell and Zermelo on propositions. One interesting difference between Russell and Zermelo lies in the nature of the individuals they admit. Russell's are nonmathematical, mathematical objects being higher-type objects in his system. Zermelo countenances individuals of every sort, but his focus is mathematical propositions whose individual constituents are numbers or sets. As for relational constituents, we should think of $R, S, \ldots$ below as irreducibly mathematical relations: triadic R might hold between group elements ϕ, ψ, and χ just in case $\phi \circ \psi = \chi$, dyadic S might be membership, and so forth. Also, Zermelo's propositions have truth values only relative to interpretations of constituent fundamental relations. They are not timelessly true or false, as are Russell's propositions.

§3. Permuting domain elements.

3.1. Metalogical notions. In *1932* Zermelo introduces a sequence of metalogical notions in quick succession, where each notion is defined in terms of truth partitions.

> If Mod($\mathcal{A}$) is the set $\mathfrak{M}$ of all matrices, then $\mathcal{A}$ is satisfied by all possible truth partitions and is 'absolutely true' for the assumed basis $\mathfrak{G}_{\mathfrak{D},\Sigma}$. On the other hand, if Mod($\mathcal{A}$) is empty, then it is satisfied by no partition and is hence 'absolutely false' or 'contradictory' for our basis. In every other case, proposition $\mathcal{A}$ is 'satisfiable' or 'contradiction-free'. If Mod($\mathcal{A}$) is a subset of Mod($\mathcal{B}$), then $\mathcal{B}$ is satisfied by every assignment that satisfies $\mathcal{A}$; we say that $\mathcal{B}$ 'follows from $\mathcal{A}$', being 'deducible' or 'provable' from it. If Mod($\mathcal{A}$) is identical with Mod($\mathcal{B}$), then the two propositions are 'equivalent', i.e., simultaneously true or false. If the intersection of Mod($\mathcal{A}$) and Mod($\mathcal{B}$) is empty, then $\mathcal{A}$ and $\mathcal{B}$ are 'irreconcilable' or mutually contradictory—otherwise they are 'reconcilable'. (*Zermelo 1932*, p. 88 [different symbols in original])

Zermelo's definitions presuppose an explanation, which he does not supply, of what it is for a matrix—in the general case, a p-tuple of order-$|\mathfrak{D}|$ Boolean tensors—to make proposition $\mathcal{A}$ true. (A tensor is the multidimensional generalization of vector and matrix.)

The passage just quoted opens with an explicit reference to a set of homogeneous order-κ Boolean matrices. So relativisation to basis $\mathfrak{G}_{\mathfrak{D},\Sigma}$ such that $|\mathfrak{D}| = \kappa$ and signature $\Sigma = \langle R \rangle$ is being assumed. Similarly, each of the other metalogical properties characterised in the quoted passage involves implicit domain and signature parameters. Thus we are presented with characterizations of 'satisfiable with respect to domain $\mathfrak{D}$

and signature Σ', 'reconcilable with respect to $\mathfrak{D}$ and Σ', and so forth. Domain relativity is obvious in the case of the metalogical property Zermelo introduces next.

In the same paragraph Zermelo presents definitions of symmetry and categoricity (see §3.3 below). This proximity suggests two things. First, despite never being fully integrated within it, Zermelo's theory of symmetric propositions constitutes a part of the theory of systems of infinitely long propositions. Second, Zermelo sees symmetry and categoricity as metalogical properties of propositions on a par with validity, satisfiability, and so forth. In particular, to the extent that the symmetry concept will be Zermelo's analysis of 'general proposition', logic is 'self-characterising', which is in keeping with Zermelo's description of logic as a 'reflexive science'. Incidentally, in *1986* Tarski states that an analysis of metalogical notions has a place in logic textbooks. Yet no edition of *Tarski 1941* incorporates analytical issues.

It is natural to ask for nonrelativised versions of the various metalogical properties Zermelo introduces. A nonrelativised notion of satisfiability is obvious: proposition $\mathcal{A}$ will be satisfiable in this nonrelativised sense just in case it is satisfiable relative to some domain $\mathfrak{D}$ and some signature Σ. A nonrelativised notion of absolute truth is ultimately unproblematic but does raise issues regarding the meaningfulness of domain-relativisation since, on the most natural proposal, any proposition absolutely true relative to any system $\mathcal{H}_{\mathfrak{D},\Sigma}$ is absolutely true in the nonrelativised sense.

3.2. Regarding the quotation from *Zermelo 1932* that opens §2 of ZAGP. Zermelo's use of the term 'urelement' within his theory of systems of infinitely long propositions is unfortunate. For what it suggests is (1) nonsets being used as (2) the basis for erecting some hierarchy of more structured mathematical objects. On both counts, it is the members of $\mathfrak{G}_{\mathfrak{D},\Sigma}$ that serve in this role as in ($\star$) in ZAGP, but Zermelo sometimes refers to the elements of $\mathfrak{D}$ as urelements, which suggests a cumulative hierarchy of urelements and sets instead. In the quoted passage from *Zermelo 1932* it would be preferable to refer to $a, b, c, \ldots$ simply as domain elements, as is done in *Zermelo 1932* paragraph 2.

3.3. Categorical propositions. Although it plays no role in the argument of ZAGP, we mention a second notion that figures prominently in Zermelo's development of his theory of symmetric propositions. A *categorical proposition* $\mathcal{A}$ *over domain* $\mathfrak{D}$ *and signature* Σ is one such that (1) $\mathrm{Mod}(\mathcal{A})$ is nonempty and permutation-invariant and (2) no nonempty proper subset of $\mathrm{Mod}(\mathcal{A})$ is permutation-invariant. Apparently any categorical proposition is both satisfiable and symmetric. In *Zermelo 1935* it is demonstrated that any symmetric proposition is expressible, uniquely up to logical equivalence, as the disjunction of categorical propositions. (Zermelo's terminology harkens to Kant's distinction between categorical and disjunctive judgements.) Categorical propositions, or their equivalence classes, serve as atoms of the complete atomic Boolean algebra of symmetric propositions over a given $\mathfrak{D}$ and Σ. Zermelo's emphasis upon the subcategory of categorical propositions, defined semantically but not structurally, provides another, aesthetic, reason to prefer his semantic notion of symmetry.

§4. The general character of mathematical axioms. Regarding Russell's radical take on Kant's Indifference Principle we mention a certain parallel within the philosophy of science. For instance, Popper requires that general laws of scientific theories mention no particular entities (see *Popper 1959*), and Hempel and Oppenheim place a similar restriction on "fundamental laws" at least (see *Hempel and Oppenheim 1948*).

§5. Mathematical axioms and symmetry. We begin with a disclaimer. Kant holds that, in addition to being general, any axiom of geometry must be synthetic and a priori, terms that Zermelo, not surprisingly, never invokes. (As a mathematician with a hand in physics, Zermelo would reject Kant's theory of geometry and space on general grounds.) Further, Kant holds that arithmetic has no axioms: those arithmetic propositions that qualify as synthetic, e.g. $\lceil 7 + 5 = 12 \rceil$, happen to be singular, whereas appropriately general propositions such as $\lceil \forall abc\,(a = b \Rightarrow a{+}c = b{+}c) \rceil$ are analytic, writes Kant. (Since Dedekind's axioms have landmark status for any mathematician with an interest in foundations, Zermelo would not follow Kant here either.) If anything, Zermelo's philosophy of mathematics, as opposed to his philosophy of logic, is anti-Kantian. Kant's philosophy of mathematics is derived from a certain conception of human nature, specifically, human intuition, of which space (geometry) and time (arithmetic) constitute the elemental structure. In contrast, Zermelo's platonism is grounded in a conception of the mathematical realm as independent of human cognition.

Note that the structuralist reading of $\mathcal{A}^*$ of ZAGP is precisely what our alternative construal of $\lceil 5 + 7 = 12 \rceil$ as symmetric proposition (†) of ZAGP demands: the unique domain element $\mathfrak{a}$ such that Five($\mathfrak{a}$) holds is the five-successor of the current 0, whatever that happens to be. But reading $\lceil 5{+}7 = 12 \rceil$ as (†) might appear problematic for a different reason. Namely, when Zermelo announces that mathematical axioms must be symmetric, he surely intends with Kant to rule out the likes of $\lceil 5{+}7 = 12 \rceil$ as a possible axiom for number theory. (That, we are convinced, is what motivates Zermelo's several statements of Strong Generality cited in §4 of ZAGP.) So how is one to reconcile symmetric (†) with Strong Generality?

The answer lies in noting that (†) of ZAGP comes out symmetric because one will have defined Five( ) in such a way that its holding of some

$\mathfrak{a} \in \mathfrak{D}$ modulo $M \in \mathfrak{M}_{\mathfrak{D}}^{1,1,1,3}$ implies that analogues of (1′), (2′), and (3′) of ZAGP are true of M, and similarly for Seven( ) and Twelve( ). In contrast, if Five( ), Seven( ), and Twelve( ) do not impose such structural requirements, then (†) will not be symmetric in general. For example, where $a, b, c \in \mathfrak{D}$ and $\Sigma = \langle F, S, T, + \rangle$ with F, S, T monadic and $+$ triadic, let $\ulcorner 5+7=12 \urcorner$ be construed not as (†) but, instead, nonsymmetrically as

$$(Fa \wedge \prod_{\mathfrak{a}\neq a} \neg F\mathfrak{a}) \wedge (Sb \wedge \prod_{\mathfrak{a}\neq b} \neg S\mathfrak{a}) \wedge (Tc \wedge \prod_{\mathfrak{a}\neq c} \neg F\mathfrak{a}) \wedge +(a, b, c).$$

In other words, $\ulcorner 5+7=12 \urcorner$ qua (†) is symmetric, and hence a candidate axiom, only if understood so as to imply the very structure imposed on M by the simpler $\mathcal{A}^*$.

§6. Toward an analysis of 'general proposition'. In addition to being general, an axiom is necessarily synthetic according to Kant; but syntheticity concerns the manner in which an axiom would be verified and hence is not among its logical features. So generality would seem to be the foremost logical feature of any axiom according to him.

§7. Symmetric propositions and first-order logic. The reference at the bottom of the three paragraphs before the end of §7 of ZAGP is to Corollary 8.3 and Theorem 9.6 of *Taylor unpublished*.

7.1. An expansion theorem for first-order logic. In accordance with the approach described in *McGee 1996*, we shall think of standard formula $\Phi(\chi_1, \ldots, \chi_k)$ of $\mathcal{L}^{\mathrm{FOL}}_{\mathfrak{D},\Sigma_=}$—that is, one containing no occurrences of individual constants—as defining a certain operation $\circledast_{\Phi(\chi_1,\ldots,\chi_k)}$ on $\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ whose value for any $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ is a subset of $\mathfrak{D}^k$. Namely, given $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, we set

$$(*)\ \circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\tau) =: \left\{ \langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k \rangle \in \mathfrak{D}^k \;\middle|\; \tau \models [\Phi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}} \right\}.$$

We can then prove

THEOREM 7.1. *Let $\Phi(\chi_1, \ldots, \chi_k)$ be a standard formula of $\mathcal{L}^{\mathrm{FOL}}_{\mathfrak{D},\Sigma_=}$. Then $\circledast_{\Phi(\chi_1,\ldots,\chi_k)} : \mathfrak{T}^{\vec{n}}_{\mathfrak{D}} \mapsto \wp(\mathfrak{D}^k)$ satisfies*

$$\circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\pi(\tau)) = \pi(\circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\tau))$$

for all $\pi \in S_{\mathfrak{D}}$ and all $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$.

PROOF. The proof proceeds by induction on the complexity of formula $\Phi(\chi_1, \ldots, \chi_k)$. First, suppose that $\Phi(\chi_1, \ldots, \chi_k)$ is atomic $\mathbf{R}_\ell \chi_{j_1} \ldots \chi_{j_{n_\ell}}$ with $1 \leq j_1, \ldots, j_{n_\ell} \leq k$. Then $[\Phi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$ is $R_\ell \mathfrak{a}_{j_1} \ldots \mathfrak{a}_{j_{n_\ell}}$ so that, for any $\pi \in S_{\mathfrak{D}}$ and any $\tau =: \langle f_1, \dots, f_p\rangle \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$, we can write

$$\begin{aligned} &\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\pi(\tau)) \\ =&\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}\end{aligned}$$

by Definition $*$

$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid [\pi(f_\ell)](\mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_{n_\ell}}) = 1\}$$

since $[\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$ is $R_\ell \mathfrak{a}_{j_1}\dots\mathfrak{a}_{j_{n_\ell}}$

$$\begin{aligned} =&\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \langle \mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_{n_\ell}}\rangle \in [\pi(f_\ell)]^{-1}(1)\} \\ =&\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid [\pi(f_\ell)](\mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_{n_\ell}}) = 1\} \\ =&\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid f_\ell(\pi^{-1}(\mathfrak{a}_{j_1}),\dots,\pi^{-1}(\mathfrak{a}_{j_{n_\ell}})) = 1\}\end{aligned}$$

by the definition of function $\pi(f_\ell)$

$$\begin{aligned} =&\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \langle \pi^{-1}(\mathfrak{a}_{j_1}),\dots,\pi^{-1}(\mathfrak{a}_{j_{n_\ell}})\rangle \in f_\ell^{-1}(1)\} \\ =&\{\pi(\pi^{-1}(\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle)) \in \mathfrak{D}^k \mid \langle \pi^{-1}(\mathfrak{a}_{j_1}),\dots,\pi^{-1}(\mathfrak{a}_{j_{n_\ell}})\rangle \in f_\ell^{-1}(1)\} \\ =&\pi(\{\pi^{-1}(\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle) \in \mathfrak{D}^k \mid \pi^{-1}(\langle \mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_{n_\ell}}\rangle) \in f_\ell^{-1}(1)\}) \\ =&\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \langle \mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_{n_\ell}}\rangle \in f_\ell^{-1}(1)\})\end{aligned}$$

since every $\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k$ is $\pi^{-1}(\langle \mathfrak{b}_1,\dots,\mathfrak{b}_k\rangle)$ for some $\mathfrak{b}_1,\dots,\mathfrak{b}_k \in \mathfrak{D}$

$$\begin{aligned} =&\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid f_\ell(\mathfrak{a}_{j_1},\dots,\mathfrak{a}_{j_{n_\ell}}) = 1\}) \\ =&\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\})\end{aligned}$$

since $[\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$ is $R_\ell \mathfrak{a}_{j_1}\dots\mathfrak{a}_{j_{n_\ell}}$

$$=\pi(\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\tau))$$

by Definition $*$ again.

If $\Phi(\vec{\chi})$ is $\chi_{j_1} \doteq \chi_{j_2}$ with $1 \le j_1, j_2 \le k$, then $[\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$ is $\mathfrak{a}_{j_1} = \mathfrak{a}_{j_2}$, where $\mathfrak{a}_{j_1}$ and $\mathfrak{a}_{j_2}$, so that, for any $\tau =: \langle f_1,\dots,f_p\rangle \in \mathfrak{T}_{\mathfrak{D}}^{\vec{n}}$ and any $\pi \in S_{\mathfrak{D}}$, we can write

$$\begin{aligned} &\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\pi(\tau)) \\ =&\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}\end{aligned}$$

by Definition $*$

$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid [\pi(\mathrm{Id})](\mathfrak{a}_{j_1},\mathfrak{a}_{j_2}) = 1\}$$

since $[\Phi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$ is $\mathfrak{a}_{j_1} = \mathfrak{a}_{j_2}$

$$\begin{aligned}
&=\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \mathrm{Id}(\mathfrak{a}_{j_1}, \mathfrak{a}_{j_2}) = 1\}\\
&=\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \mathfrak{a}_{j_1} \text{ is } \mathfrak{a}_{j_2}\}\\
&=\pi(\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \mathfrak{a}_{j_1} \text{ is } \mathfrak{a}_{j_2}\})\\
&=\pi(\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \mathrm{Id}(\mathfrak{a}_{j_1}, \mathfrak{a}_{j_2}) = 1\})\\
&=\pi(\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Phi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\})
\end{aligned}$$

since $[\Phi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$ is $\mathfrak{a}_{j_1} = \mathfrak{a}_{j_2}$

$$=\pi(\circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\tau))$$

by Definition $*$ again.

(In effect, $\circledast_{\Phi(\chi_1,\ldots,\chi_k)}$ is a constant function in this case.)

As the first of three inductive subcases, suppose that $\Phi(\chi_1, \ldots, \chi_k)$ is of the form $\dot\neg(\Psi(\chi_1, \ldots, \chi_k))$, where $\Psi(\chi_1, \ldots, \chi_k)$ defines operation $\circledast_{\Psi(\chi_1,\ldots,\chi_k)}$ satisfying Definition $*$. For all $\pi \in S_{\mathfrak{D}}$ and all $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, we have

$$\begin{aligned}
&\circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\pi(\tau))\\
=&\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\dot\neg(\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k))]^+_{\mathfrak{D}}\}
\end{aligned}$$

by Definition $*$ since $\Phi(\chi_1, \ldots, \chi_k)$ is $\dot\neg(\Psi(\chi_1, \ldots, \chi_k))$

$$=\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models \neg([\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}})\}$$

by clause 3 in the definition of canonical expansion given in ZAGP

$$\begin{aligned}
&=\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \not\models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}\\
&=\overline{\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}}\\
&=\overline{\circledast_{\Psi(\chi_1,\ldots,\chi_k)}(\pi(\tau))}
\end{aligned}$$

by Definition $*$ again

$$=\overline{\pi(\circledast_{\Psi(\chi_1,\ldots,\chi_k)}(\tau))}$$

by induction hypothesis

$$\begin{aligned}
&=\overline{\pi(\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\})}\\
&=\pi(\overline{\{\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots, \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}})
\end{aligned}$$

since π permutes $\mathfrak{D}^k$

$$
\begin{aligned}
&=\pi(\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \not\models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\})\\
&=\pi(\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models \neg([\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}})\})\\
&=\pi(\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\dot\neg(\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k))]^+_{\mathfrak{D}}\})
\end{aligned}
$$

by clause 3 in the definition of canonical expansion given in ZAGP

$$=\pi(\circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\tau))$$

by Definition $*$ again since $\Phi(\chi_1, \ldots , \chi_k)$ is $\dot\neg(\Psi(\chi_1, \ldots , \chi_k))$.

Let $\Phi(\chi_1, \ldots , \chi_k)$ be of the form $(\Psi(\chi_1, \ldots , \chi_k)) \dot\vee (\Xi(\chi_1, \ldots , \chi_k))$, where $\Psi(\chi_1, \ldots , \chi_k)$ and $\Xi(\chi_1, \ldots , \chi_k))$ define operations $\circledast_{\Psi(\chi_1,\ldots,\chi_k)}$ and $\circledast_{\Xi(\chi_1,\ldots,\chi_k)}$, respectively, satisfying Definition $*$. For all $\pi \in S_{\mathfrak{D}}$ and all $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, we can write

$$
\begin{aligned}
&\circledast_{\Phi(\chi_1,\ldots,\chi_k)}(\pi(\tau))\\
=&\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [(\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)) \dot\vee (\Xi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k))]^+_{\mathfrak{D}}\}
\end{aligned}
$$

by Definition $*$ since $\Phi(\chi_1, \ldots , \chi_k)$ is $(\Psi(\chi_1, \ldots , \chi_k)) \dot\vee (\Xi(\chi_1, \ldots , \chi_k))$

$$=\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models ([\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}) \vee ([\Xi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}})\}$$

by clause 4 in the definition of canonical expansion given in ZAGP

$$
\begin{aligned}
=&\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}} \text{ or}\\
&\qquad\qquad \pi(\tau) \models [\Xi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}\\
=&\{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Psi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\} \cup\\
&\qquad\qquad \{\langle \mathfrak{a}_1, \ldots , \mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Xi(\boldsymbol{\mathfrak{a}}_1, \ldots , \boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\}\\
=&\circledast_{\Psi(\chi_1,\ldots,\chi_k)}(\pi(\tau)) \cup \circledast_{\Xi(\chi_1,\ldots,\chi_k)}(\pi(\tau))
\end{aligned}
$$

by Definition $*$ again

$$=\pi(\circledast_{\Psi(\chi_1,\ldots,\chi_k)}(\tau)) \cup \pi(\circledast_{\Xi(\chi_1,\ldots,\chi_k)}(\tau))$$

by induction hypothesis

$$=\pi(\circledast_{\Psi(\chi_1,\dots,\chi_k)}(\tau) \cup \circledast_{\Xi(\chi_1,\dots,\chi_k)}(\tau))$$
$$=\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\} \cup \{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Xi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\})$$
$$=\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}} \text{ or } \tau \models [\Xi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\})$$
$$=\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models ([\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}) \vee ([\Xi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}})\})$$
$$=\pi(\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \tau \models [(\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)) \dot{\vee} (\Xi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k))]^+_{\mathfrak{D}}\})$$

by clause 4 in the definition of canonical expansion given in ZAGP

$$=\pi(\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\tau))$$

by Definition $*$ since $\Phi(\chi_1,\dots,\chi_k)$ is $(\Psi(\chi_1,\dots,\chi_k)) \dot{\vee} (\Xi(\chi_1,\dots,\chi_k))$.

Finally, suppose that $\Phi(\chi_1,\dots,\chi_k)$ is of the form $\exists\upsilon(\Psi(\chi_1,\dots,\chi_k,\upsilon))$, where $\Psi(\chi_1,\dots,\chi_k,\upsilon)$ defines operation $\circledast_{\Psi(\chi_1,\dots,\chi_k,\upsilon)}$. For all $\pi \in S_{\mathfrak{D}}$ and all $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, we can write

$$\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\pi(\tau))$$
$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\exists\upsilon(\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\upsilon))]^+_{\mathfrak{D}}\}$$

by Definition $*$ since $\Phi(\chi_1,\dots,\chi_k)$ is $\exists\upsilon(\Psi(\chi_1,\dots,\chi_k,\upsilon))$

$$=\left\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \;\middle|\; \pi(\tau) \models \bigvee\{[\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\boldsymbol{\mathfrak{b}})]^+_{\mathfrak{D}} \mid \mathfrak{b} \in \mathfrak{D}\}\right\}$$

by clause 5 in the definition of canonical expansion given in ZAGP

$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi(\tau) \models [\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\boldsymbol{\mathfrak{b}})]^+_{\mathfrak{D}} \text{ for some } \mathfrak{b} \in \mathfrak{D}\}$$
$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \langle \mathfrak{a}_1,\dots,\mathfrak{a}_k,\mathfrak{b}\rangle \in \circledast_{\Psi(\chi_1,\dots,\chi_k,\upsilon)}(\pi(\tau)) \text{ for some } \mathfrak{b} \in \mathfrak{D}\}$$
$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \langle \mathfrak{a}_1,\dots,\mathfrak{a}_k,\mathfrak{b}\rangle \in \pi(\circledast_{\Psi(\chi_1,\dots,\chi_k,\upsilon)}(\tau)) \text{ for some } \mathfrak{b} \in \mathfrak{D}\}$$

by induction hypothesis

$$=\{\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle \in \mathfrak{D}^k \mid \pi^{-1}(\langle \mathfrak{a}_1,\dots,\mathfrak{a}_k,\mathfrak{b}\rangle) \in \circledast_{\Psi(\chi_1,\dots,\chi_k,\upsilon)}(\tau) \text{ for some } \mathfrak{b} \in \mathfrak{D}\}$$

since π is a permutation

$$=\{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle\in\mathfrak{D}^k \mid \tau\models[\Psi(\mathrm{c}_{\pi^{-1}(\mathfrak{a}_1)},\dots,\mathrm{c}_{\pi^{-1}(\mathfrak{a}_k)},\mathrm{c}_{\pi^{-1}(\mathfrak{b})})]^+_{\mathfrak{D}} \text{ for some } \mathfrak{b}\in\mathfrak{D}\}$$

where, with $1\le i\le k$, we write $\mathrm{c}_{\pi^{-1}(\mathfrak{a}_i)}$ for the individual constant of language $\mathfrak{L}^{\mathrm{FOL}}_{\mathfrak{D},\Sigma_=}$ that denotes domain element $\pi^{-1}(\mathfrak{a}_i)$, and similarly in the case of $\mathrm{c}_{\pi^{-1}(\mathfrak{b})}$

$$=\{\pi(\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle)\in\mathfrak{D}^k \mid \tau\models[\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\boldsymbol{\mathfrak{b}})]^+_{\mathfrak{D}} \text{ for some } \mathfrak{b}\in\mathfrak{D}\}$$
$$=\pi(\{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle\in\mathfrak{D}^k \mid \tau\models[\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\boldsymbol{\mathfrak{b}})]^+_{\mathfrak{D}} \text{ for some } \mathfrak{b}\in\mathfrak{D}\})$$
$$=\pi\Big(\Big\{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle\in\mathfrak{D}^k \Bigm| \tau\models\bigvee\{[\Psi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\boldsymbol{\mathfrak{b}})]^+_{\mathfrak{D}}\mid\mathfrak{b}\in\mathfrak{D}\}\Big\}\Big)$$
$$=\pi(\{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle\in\mathfrak{D}^k \mid \tau\models[\exists\upsilon(\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k,\upsilon))]^+_{\mathfrak{D}}\})$$

by clause 5 in the definition of canonical expansion given in ZAGP

$$=\pi(\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\tau))$$

by Definition $*$ again since $\Phi(\chi_1,\dots,\chi_k)$ is $\exists\upsilon(\Psi(\chi_1,\dots,\chi_k,\upsilon))$.

$\dashv$

THEOREM 7.2 (Expansion Theorem for $\mathfrak{L}^{\mathrm{FOL}}_{\mathfrak{D},\Sigma_=}$). *Let $\Phi(\chi_1,\dots,\chi_k)$ be a standard formula of $\mathfrak{L}^{\mathrm{FOL}}_{\mathfrak{D},\Sigma_=}$ and let $\mathfrak{a}_1,\dots,\mathfrak{a}_k\in\mathfrak{D}$. Then, for any $\pi\in S_{\mathfrak{D}}$ and any $\tau\in\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, we have*

$$(\dagger)\qquad \tau\models[\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\Longleftrightarrow\pi(\tau)\models[\Phi(\mathrm{c}_{\pi(\mathfrak{a}_1)},\dots,\mathrm{c}_{\pi(\mathfrak{a}_k)})]^+_{\mathfrak{D}}.$$

PROOF. It is sufficient to establish that

$$\tau\models[\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}$$
$$\Longrightarrow\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle\in\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\tau)$$
$$\Longrightarrow\pi(\langle\mathfrak{a}_1,\dots,\mathfrak{a}_k\rangle)\in\pi(\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\tau))$$
$$\Longrightarrow\langle\pi(\mathfrak{a}_1),\dots,\pi(\mathfrak{a}_k)\rangle\in\circledast_{\Phi(\chi_1,\dots,\chi_k)}(\pi(\tau))$$

by Theorem 7.1

$$\Longrightarrow\pi(\tau)\models[\Phi(\mathrm{c}_{\pi(\mathfrak{a}_1)},\dots,\mathrm{c}_{\pi(\mathfrak{a}_k)})]^+_{\mathfrak{D}}.$$

$\dashv$

COROLLARY 7.3. *Suppose that standard formula $\Phi(\chi_1,\dots,\chi_k)$ is a sentence of $\mathfrak{L}^{\mathrm{FOL}}_{\mathfrak{D},\Sigma_=}$ and let $\mathfrak{a}_1,\dots,\mathfrak{a}_k\in\mathfrak{D}$. Then $[\Phi(\boldsymbol{\mathfrak{a}}_1,\dots,\boldsymbol{\mathfrak{a}}_k)]^+_{\mathfrak{D}}\in\mathrm{Sym}_{\mathfrak{D},\Sigma_=}$.*

PROOF. If $\Phi(\chi_1, \ldots, \chi_k)$ is a sentence, then $\Phi(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$ is identical with $\Phi(\mathrm{c}_{\pi(\mathfrak{a}_1)}, \ldots, \mathrm{c}_{\pi(\mathfrak{a}_k)})$ in Equivalence †. It follows that the model set of the expansion of $\Phi(\chi_1, \ldots, \chi_k)$, modulo saturation by an arbitrary k-tuple of individual constants, is permutation-invariant. ⊣

7.2. Did Zermelo anticipate Tarski by decades? What we have called Tarski's Thesis—the claim that logical terms over fixed domain are precisely those that are permutation-invariant—was first publicly articulated by Tarski in his 1966 London lecture published as *Tarski 1986*. It is shown in *Taylor 2008* that Zermelo's analysis of general proposition, published in *Zermelo 1932*, is equivalent to Tarski's notion of logical term. This would suggest that Zermelo anticipated Tarski by a few decades. Is this correct? We think that the answer is yes and no.

In his 1966 lecture Tarski directs his listener to *Lindenbaum and Tarski 1934*, wherein Theorem 1 there may be taken to say that all the higher-order notions introduced in *Principia* are invariant under arbitrary permutations of the domain of individuals. (A detailed proof of that result was, much later, carried out in *Mostowski 1948*, according to Tarski.) That is not yet to formulate any claim regarding the nature of logical notions, of course. In any case, with respect to publication, Zermelo would seem to have anticipated Tarski by a considerable period. However, the story is more complicated.

Zermelo spent several weeks at the University of Warsaw as visiting lecturer during the summer of 1929 (see *Ebbinghaus with Peckhaus 2007*). Talk of domain permutations would have been in the air in Warsaw during Zermelo's visit. There is nothing concerning domain permutations in the drafts of Zermelo's nine Warsaw lectures nor in the hundreds of pages of notes in *Farber 1926–27*, indicating that, most likely, Zermelo was not thinking along these lines prior to his stay in Warsaw. So direct influence of Warsaw logicians, including Tarski, upon Zermelo is a possibility. Direct influence might even explain the elliptical character of Zermelo's writing concerning general propositions. Is it possible that a more explicit presentation of his analysis would have revealed the derivative nature of his own use of permutation-invariance? Our own view is that Zermelo's accomplishment with the concept of symmetric proposition would have been undiminished if he had written more explicitly, even citing Polish sources. However, Zermelo may have seen things differently.

§8. Unanalysability of the fundamental concept of generality. Two early readers of ZAGP suggested that, in the passages quoted, Zermelo's target is second-order quantification only. That would be to assimilate Zermelo's remark regarding what "underlies every mathematical observation" to the commonplace view expressed, for instance, by A. Church when he writes:

> A certain absolute notion of ALL propositional functions of individuals is presupposed in classical mathematics, especially classical analysis. (*Church 1956*, p. 326, footnote 535)

However, we would insist that Zermelo's focus in *Zermelo 1931* is less narrow. This is clear, in our view, based on the text of *Zermelo 1931* itself and on its alleged source in *Husserl 1922*, where the notion of generality at issue is the broadest possible.

Regarding the single quotation from *Husserl 1922*, we have translated the German *Moment* as 'dependent element' as on page 82 in *Dummett 1993*.

§9. Conclusion.

References

Church, A. 1956. *Introduction to Mathematical Logic*, Vol. I, Princeton, New Jersey: Princeton University Press.

Dummett, M. 1993. *Origins of Analytical Philosophy*, Cambridge, Massachusetts: Harvard University Press.

Ebbinghaus, H.-D. 'Zermelo: definiteness and the universe of definable sets', *History and Philosophy of Logic*, **24**, 197–219.

Ebbinghaus, H.-D. in cooperation with Peckhaus, V. 2007. *Ernst Zermelo: An Approach to His Life and Work*, Berlin: Springer-Verlag.

Farber, M. 1926–27. Undated handwritten notes of conversations with Zermelo concerning the philosophy of logic and mathematics (unpublished). Housed in the Marvin Farber Papers on Philosophy and Phenomenology (22/5F/768), University Archives, State University of New York at Buffalo (folders 33.10–11).

Farber, M. 1927. *The Foundations of Logic: Studies Concerning the Structure and Function of Logic*, Chapters I–III (unpublished). Housed in the Marvin Farber Papers on Philosophy and Phenomenology (22/5F/768) at University Archives, State University of New York at Buffalo (folder 28.12).

Farber, M. 1928. *Phenomenology as a Method and as a Philosophical Discipline*, *University of Buffalo Studies*, **6**. Farber's doctoral dissertation (Harvard, 1925). Zermelo's doctrine of substrates is discussed on pages 90–91.

Farber, M. 1929. 'Theses concerning the foundations of logic', *Philosophical Review*, **38**, 219–31.

Farber, M. 1930. 'The method of deduction and its limitations', *Journal of Philosophy*, **27**, 505–15.

Hempel, C. G. 1965. *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science*, New York: Free Press.

Hempel, C. G. and Oppenheim, P. 1948. 'Studies in the logic of explanation', *Philosophy of science*, **15**, 135–75. Reprinted in *Hempel 1965*, 245–90.

Husserl, E. 1922. *Logische Untersuchungen*, vol. II: *Untersuchungen zur Phänomenologie und Theorie der Erkenntnis*, part I, 3rd ed., Halle: Max Niemeyer.

McGee, V. 1996. 'Logical operations', *Journal of Philosophical Logic*, **25**, 567–80.

Mostowski, A. 1948. *Logica matematyczna: Kurs uniwersytecki*, Warsaw and Wroclaw: Monografie matematyczne, vol. 18.

Popper, K. 1959. *The Logic of Scientific Discovery*, London: Hutchinson.

Tarski, A. 1941. *Introduction to Logic and to the Methodology of the Deductive Sciences*, Olaf Helmer, trans., New York: Oxford University Press.

Tarski, A. 1986. 'What are logical notions?', *History and Philosophy of Logic*, **7**, 143–54. Based on lectures delivered in London and Buffalo in 1966 and 1973, respectively. Edited with an introduction by John Corcoran.

Taylor, R. G. 2009. 'Zermelo's analysis of "general proposition" ', *History and Philosophy of Logic*, **30**, 141–55.

Taylor, R. G. Unpublished. 'Symmetric propositions and the bounds of logic'.

Van Dalen, D. and Ebbinghaus, H.-D. 2000. 'Zermelo and the Skolem paradox', *Bulletin of Symbolic Logic*, **6**, 145–61.

Van Heijenoort, J. 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Cambridge, Massachusetts: Harvard University Press.

Zermelo, E. 1908. 'Untersuchungen über die Grundlagen der Mengenlehre I', *Mathematische Annalen*, **65**, 261–81. Translated as 'Investigations in the foundations of set theory' in *Van Heijenoort 1967*, 199–215.

Zermelo, E. 1921. 'Thesen über das Unendliche in der Mathematik', one-page typescript dated 17th July 1921 and now housed in Nachlaß Ernst Zermelo, Universitätsarchiv Freiburg C 129. German original and English translation appear in *Van Dalen and Ebbinghaus 2000*.

Zermelo, E. 1931. 'Sätze über geschlossene Bereiche', two-page typescript (Universitätsarchiv Freiburg C 129) from around 1931 but bearing no date. Appears as an appendix to *Ebbinghaus 2004*.

Zermelo, E. 1932. 'Über Stufen der Quantifikation und die Logik des Unendlichen', *Jahresbericht der deutschen Mathematikervereinigung (Angelegenheiten)*, **41**, 85–88.

Zermelo, E. 1935. 'Grundlagen einer allgemeinen Theorie der mathematischen Satzsysteme (erste Mitteilung)', *Fundamenta mathematicæ*, **25**, 136–46.

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